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A new method for eigenvalue estimates for Dirac operators on certain manifolds with S^k-symmetry ☆

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Abstract

We prove estimates for the first nonnegative eigenvalue of the Dirac operator on certain manifolds with $SO(k+1)$-symmetry in terms of geometric properties of the manifold. For the proof we employ an abstract technique which is new in this context and may apply to other cases of manifolds as well.

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1. Introduction

The Dirac operator on a closed n-dimensional Riemannian spin manifold M^n is an essentially self-adjoint operator with discrete spectrum which can be calculated only in some particular cases (e.g., for spheres [24], flat tori [13], complex projective spaces [9,10,23], and spherical space forms [4]).

In general, only estimates for the eigenvalues can be obtained. The first general estimate in the case of a manifold with positive scalar curvature was proved in [14]. In [18] a lower bound for the eigenvalues of the Dirac operator on manifolds diffeomorphic to S^{k+1} with an isometric effective $SO(k+1)$-action was established. Analogous results were derived in [19] for fibrations over S^1 and in [1] (see also [2,3]) for certain families of fiber bundles M_n with fiber S^1 over closed m-dimensional spin manifolds B.

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In the present paper we use a novel operator theoretic technique in order to prove a lower bound for the first nonnegative eigenvalue of the Dirac operator on warped products with fiber S^k over m-dimensional manifolds B with boundary ∂B for arbitrary k and m. Our main result is an estimate for eigenvalues of weight μ (see Definition 7.1 below) of the Dirac operator D_M on a warped product with boundary $M = B \times_f S^k$ (see Definition 2.3). In particular, we prove that the spectrum of D_M is symmetric with respect to 0, that D_M has trivial kernel (which need not be the case in general, see, e.g., [6,22]), and that the smallest nonnegative eigenvalue of D_M satisfies the estimate

$$\lambda_0 \geqslant \frac{k}{2 f_{\max}}$$

where $f_{\max} := \max\{f(x)\colon x \in B\}$ is the maximum of the warpe function f. This result is sharp since in a limiting case it coincides with a well-known upper bound for the first positive eigenvalue of the Dirac operator on an immersed Riemannian manifold derived in [5,8].

Warped products with boundary are manifolds with effective $SO(k+1)$-action which have only two types of orbits, namely k-spheres and fixed points. For $m = 1$ examples are surfaces of revolution or the manifolds considered in [18], for $m > 1$ examples are spheres with the standard metric or m-dimensional ellipsoids with $SO(k+1)$-symmetry. More exactly, if $f : B \to \mathbb{R}_0^+$ is a C^∞-function with $f|_{B_0} > 0$, $f|_{\partial B} \equiv 0$, where $B_0 := B \setminus \partial B$ denotes the interior of B, then the closure $M = B \times_f S^k$ of the warped product $M_0 = B_0 \times_f S^k = (B_0 \times S^k, m_{B_0} + f^2 m_{S^k})$ with a metric m_{B_0} on B_0 and the standard metric m_{S^k} on S^k becomes an $(m+k)$-dimensional manifold (under some additional conditions on the metric m_B on B and on f). M is called warped product with boundary.

The key idea of the proof of our eigenvalue estimate is to decompose the Dirac operator D_{M_0} into a direct sum of lower-dimensional operators D_μ having a block operator matrix representation which allows to invoke an abstract theorem on spectral gaps. More precisely, we decompose the Dirac operator D_{M_0} into a direct sum

$$\mathrm{D}_{M_0} = \bigoplus_{\mu \in \mathrm{Spec}_p(\mathrm{D}_{S^k})} \mathrm{D}_\mu$$

induced by a decomposition of the space of spinors on the fibers S^k into the eigenspaces $E(\mu, \mathrm{D}_{S^k})$, $\mu \in \mathrm{Spec}_p(\mathrm{D}_{S^k})$, of the Dirac operator on S^k. This yields a decomposition of the space of spinors of M into subspaces W_μ of spinors of weight μ, and the operators D_μ are the restrictions of D_{M_0} to the spaces W_μ. The operators D_μ lead to systems of partial differential equations and hence the ordinary differential equations techniques used in [18] cannot be applied here.

However, the operators D_μ admit so-called block operator matrix representations

$$\mathrm{D}_\mu = \begin{pmatrix} A_\mu & C \\ C^* & \widehat{A}_\mu \end{pmatrix} \tag{1.1}$$

such that there is a gap around 0 between the spectra of the diagonal operators A_μ and $\widehat{A}_\mu$. This structure allows to treat the underlying system of partial differential equations by means of an abstract operator theoretic result proved in [20] which shows that this gap, which coincides with the gap in the spectrum of the block diagonal operator matrix

$$\begin{pmatrix} A_\mu & 0 \\ 0 & \widehat{A}_\mu \end{pmatrix},$$

is retained as a gap in the spectrum of D_μ, independently of the “size” of C. This implies an estimate of the smallest nonnegative eigenvalue of each operator D_μ and, by taking the minimum over all μ, also an estimate of the smallest nonnegative eigenvalue of the operator D_M, which is the closure of D_{M_0}.

The paper is organized as follows. In Section 2 we introduce warped products with boundary and derive the decomposition of the space of spinors thereon. In Section 3 the Dirac operator D_{M_0} on a warped product $M_0 = B_0 \times_f S^k$ is expressed formally by means of Dirac operators on the basis B and the fiberwise Dirac operator D'. In Section 4 we establish the boundary conditions for the operators in (1.1) which follow from the condition that a spinor over the warped product M_0 has to have a continuous extension to M, more exactly, D_M is the closure of the restriction of D_{M_0} to all spinors over $B_0 \times_f S^k$ with this property. In Section 5, the operator theoretic result about the spectral gap of certain block operator matrices is presented. In Section 6 we establish the direct sum decomposition of the above mentioned restriction of D_{M_0} into block operator matrices satisfying the assumptions of Section 5. Applying the abstract theorem of Section 5 to the block operator matrices in this decomposition, we obtain our main result about the eigenvalues of the Dirac operator D_M in Section 7, and we show that this estimate is sharp.

2. The decomposition of the space of spinors

The manifolds considered in this paper are warped products over a basis manifold B with boundary which has a certain metric.

Definition 2.1. A Riemannian manifold B with boundary ∂B is said to have a *warped collar* if there exists a neighbourhood U of ∂B of the form

$$U = \partial B\,{}_g\!\times [0,\varepsilon) := \big(\partial B \times [0,\varepsilon),\, g^2 m_0 + \mathrm{d}t^2\big),$$

where $\varepsilon > 0$, m_0 is a metric on ∂B and $g : [0,\varepsilon) \to \mathbb{R}^+$ is the restriction of an even C^∞-function on $(-\varepsilon,\varepsilon)$ with $g(0) = 1$.

Lemma 2.2. *Let B be an m-dimensional Riemannian manifold with boundary and warped collar $U = \partial B\,{}_g\!\times [0,\varepsilon)$. Let $f : B \to \mathbb{R}_0^+$ be a C^∞-function with*

$$f|_{B_0} > 0, \qquad f|_{\partial B} \equiv 0$$

and such that on U the function $f(x,\cdot) = f$ does not depend on $x \in \partial B$ and is the restriction of an odd C^∞-function on $(-\varepsilon,\varepsilon)$ with $f'(0) = 1$.

Then the closure of the warped product $M_0 := B_0 \times_f S^k$ is an $(m+k)$-dimensional manifold denoted by $B \times_f S^k$.

Proof. A neighbourhood of $\partial B \subset B \times_f S^k$ is given by $\partial B\,{}_{\tilde g}\!\times (D_\varepsilon^{k+1}, m_f)$ where $\tilde g : D_\varepsilon^{k+1} \to \mathbb{R}^+$ is given by $\tilde g(x) := g(|x|)$ and the metric m_f on the ε-disc D_ε^{k+1} is determined by $(D_\varepsilon^{k+1}, m_f) = [0,\varepsilon) \times_f S^k$. □

Definition 2.3. Let B and f be given as in Lemma 2.2. Then the manifold $B \times_f S^k$ is called *warped product with boundary* (with basis B and warpe function f). If $B \times_f S^k$ is a spin manifold, we call $B \times_f S^k$ a *spin warped product with boundary*.

We consider M_0 as a fiber bundle over B_0 with the canonical connection and we use the notations fiber, horizontal and vertical vectors correspondingly.

For $k > 1$, we have for the first homology $H_1(B \times_f S^k) = H_1(B \times_f S^k \setminus \partial B)$ and hence the spin structures of $B \times_f S^k$ are in one-to-one correspondence with the spin structures of $B_0 \times_f S^k$ and therefore with the spin structures on B_0.

The spinor calculus for warped products with 1-dimensional basis has been discussed in [7, p. 18] and [6], the formulas for the Clifford multiplication on products have been established in [11] and more explicitly in [17]. In [15] the spinor calculus on products has been worked out in some special dimensions. In the sequel we summarize these and present the spinor calculus on warped products $M_0 = B_0 \times_f S^k$ over an m-dimensional basis B. We use the following notations:

Notation. For a Riemannian spin manifold X we denote by P_X the frame bundle over X, by Q_X a spin structure of P_X and by Σ_X the corresponding spinor bundle.

Let $\pi_1 : M_0 \to B_0$ and $\pi_2 : M_0 \to S^k$ denote the canonical projections. Then the frame bundle P_{M_0} has a reduction $\pi_1^* P_{B_0} \times_{M_0} P'$ to the structure group $SO(m) \times SO(k)$ where $P' \cong \pi_2^* P_{S^k}$ is the bundle of vertical orthogonal frames. Therefore a spin structure Q_{B_0} on B_0 induces a spin structure Q_{M_0} on M_0 by

$$Q_{M_0} = (\pi_1^* Q_{B_0} \times_{M_0} Q') \times_{\mathrm{Spin}(m)\times\mathrm{Spin}(k)} \mathrm{Spin}(m+k)$$

where is $Q' \cong \pi_2^* Q_{S^k}$ is a $\mathrm{Spin}(k)$-version of P'. We denote by $\Sigma' \to M_0$ the fiberwise spinor bundle given by $Q' \times_{\mathrm{Spin}(k)} \Delta_k$. Then $\Sigma' \cong \pi_2^* \Sigma_{S^k}$ and

$$\begin{aligned}
\Sigma_M|_{M_0} &= \pi_1^* \Sigma_{B_0} \otimes \Sigma' \cong \pi_1^* \Sigma_{B_0} \otimes \pi_2^* \Sigma_{S^k} \quad \text{for } m \text{ or } k \text{ even},\\
\Sigma_M|_{M_0} &= (\pi_1^* \Sigma_{B_0} \otimes \Sigma') \oplus (\pi_1^* \widehat{\Sigma}_{B_0} \otimes \Sigma')\\
&\cong (\pi_1^* \Sigma_{B_0} \otimes \pi_2^* \Sigma_{S^k}) \oplus (\pi_1^* \widehat{\Sigma}_{B_0} \otimes \pi_2^* \Sigma_{S^k}) \quad \text{for } m \text{ and } k \text{ odd}.
\end{aligned}$$

Here for an n-dimensional manifold X with n odd, we denote by Σ_X and $\widehat{\Sigma}_X$ the two spinor bundles given by the two representations of the n-dimensional Clifford algebra Cl_n.

The Clifford multiplication with vectors $X \in TB^m$, $Y \in TS^k$ on spinors $\psi_1 \otimes \psi_2 \in \Gamma_{M_0}(\pi_1^* \Sigma_{B_0} \otimes \pi_2^* \Sigma_{S^k})$ and $\varphi_1 \otimes \varphi_2 \in (\pi_1^* \widehat{\Sigma}_{B_0} \otimes \pi_2^* \Sigma_{S^k})$ is given by

$$\left.\begin{aligned}
X \cdot (\psi_1 \otimes \psi_2) &= X \cdot \psi_1 \otimes \psi_2\\
Y \cdot (\psi_1 \otimes \psi_2) &= \omega_{B_0} \cdot \psi_1 \otimes Y \cdot \psi_2
\end{aligned}\right. \quad \text{for } m \text{ even},$$

$$\left.\begin{aligned}
X \cdot (\psi_1 \otimes \psi_2) &= X \cdot \psi_1 \otimes \omega_{S^k} \cdot \psi_2\\
Y \cdot (\psi_1 \otimes \psi_2) &= \psi_1 \otimes Y \cdot \psi_2
\end{aligned}\right. \quad \text{for } m \text{ odd}, k \text{ even},$$

$$\left.\begin{aligned}
X \cdot (\psi_1 \otimes \psi_2 + \varphi_1 \otimes \varphi_2) &= \mathrm{i}\big(\eta^{-1}(X \cdot \varphi_1) \otimes \varphi_2 + \eta(X \cdot \psi_1) \otimes \psi_2\big)\\
Y \cdot (\psi_1 \otimes \psi_2 + \varphi_1 \otimes \varphi_2) &= \eta^{-1}(\varphi_1) \otimes Y \cdot \varphi_2 + \eta(\psi_1) \otimes Y \cdot \psi_2
\end{aligned}\right. \quad \text{for } m, k \text{ odd}.$$

Here, for an even-dimensional manifold A with local orthonormal basis $e_1, \ldots, e_a$, $\omega_A := \mathrm{i}^{a/2} e_1 \cdots e_a$ denotes the complex volume element, and for an odd-dimensional manifold A we denote by $\eta : \Sigma_A \to \widehat{\Sigma}_A$ the canonical isomorphism with $\eta(v \cdot \psi) = -v \cdot \eta(\psi)$.

The Dirac operator D_{S^k} on the sphere S^k induces an operator D' on $\Sigma' = \pi_2^* \Sigma_{S^k}$ by

$$(\mathrm{D}'\psi)(b, \cdot) = \mathrm{D}_{f(b)S^k}(\psi|_{\{b\}\times f(b)S^k}),$$

which we call fiber Dirac operator. For each eigenvalue $\mu \in \mathrm{Spec}_p(\mathrm{D}_{S^k})$ with multiplicity $r(\mu)$ we define an $r(\mu)$-dimensional vector bundle $\mathcal{E}_\mu \to B_0$ by

$$(\mathcal{E}_\mu)_b := E\left(\frac{\mu}{f(b)}, \mathrm{D}'|_{\{b\}\times f(b)S^k}\right),$$

where $E(\frac{\mu}{f(b)}, \mathrm{D}'|_{\{b\}\times f(b)S^k})$ denotes the eigenspace of the fiber Dirac operator restricted to $\{b\} \times f(b)S^k$. Let $(\sigma_{\mu,1}, \dots, \sigma_{\mu,r(\mu)})$ be an orthonormal basis of $E(\mu, \mathrm{D}_{S^k})$. Then

$$(\sigma'_{\mu,1}, \dots, \sigma'_{\mu,r(\mu)}) \quad \text{with } \sigma'_{\mu,i} := \frac{\pi_2^*\sigma_{\mu,i}}{f^{k/2}} \tag{2.1}$$

is a trivialisation of $\mathcal{E}_\mu$ by fiberwise orthonormal sections. Therefore we get a decomposition of the space of sections

$$\Gamma_{M_0}(\Sigma') = \bigoplus_{\mu\in\mathrm{Spec}_p(\mathrm{D}_{S^k})} \Gamma_{B_0}(\mathcal{E}_\mu) = \bigoplus_{\mu\in\mathrm{Spec}_p(\mathrm{D}_{S^k})} \Gamma_{B_0}\big(B_0 \times \mathbb{C}^{r(\mu)}\big)$$

as every (continuous) section $\psi \in \Gamma_{M_0}(\Sigma')$ has a decomposition $\psi = \sum_\mu \vartheta_\mu \sigma'_\mu$ with (continuous) functions ϑ_μ on B_0.

In the same way the spinor space over the warped product M_0 decomposes as

$$\Gamma_{M_0}(\Sigma_M|_{M_0}) = \bigoplus_{\mu\in\mathrm{Spec}_p(\mathrm{D}_{S^k})} \Gamma_{B_0}(\Sigma_{B_0} \otimes \mathcal{E}_\mu) \quad \text{for } m \text{ or } k \text{ even},$$

$$\Gamma_{M_0}(\Sigma_M|_{M_0}) = \bigoplus_{\mu\in\mathrm{Spec}_p(\mathrm{D}_{S^k})} \big(\Gamma_{B_0}(\Sigma_{B_0} \otimes \mathcal{E}_\mu) \oplus \Gamma_{B_0}(\widehat{\Sigma}_{B_0} \otimes \mathcal{E}_\mu)\big) \quad \text{for } m \text{ and } k \text{ odd}.$$

Definition 2.4. Let $\psi \in \Gamma_M(\Sigma_M)$ and $\mu \in \mathrm{Spec}_p(\mathrm{D}_{S^k})$. Then ψ is called a *spinor of weight* μ if

$$\begin{aligned}
&\psi|_{M_0} \in \Gamma_{B_0}(\Sigma_{B_0} \otimes \mathcal{E}_\mu) = \Gamma_{B_0}\big(\Sigma_{B_0} \otimes \mathbb{C}^{r(\mu)}\big) =: W_\mu \quad \text{if } m \text{ even},\\
&\psi|_{M_0} \in \Gamma_{B_0}\big(\Sigma_{B_0} \otimes (\mathcal{E}_\mu \oplus \mathcal{E}_{-\mu})\big) = \Gamma_{B_0}\big(\Sigma_{B_0} \otimes \mathbb{C}^{2r(\mu)}\big) =: W_\mu \quad \text{if } m \text{ odd}, k \text{ even},\\
&\psi|_{M_0} \in \Gamma_{B_0}\big(\big(\Sigma^d_{B_0} \otimes \mathcal{E}_\mu\big) \oplus \big(\Sigma^{od}_{B_0} \otimes \mathcal{E}_\mu\big)\big) = \Gamma_{B_0}\big(\big(\Sigma^d_{B_0} \otimes \mathbb{C}^{r(\mu)}\big) \oplus \big(\Sigma^{od}_{B_0} \otimes \mathbb{C}^{r(\mu)}\big)\big) =: W_\mu\\
&\quad \text{if } m, k \text{ odd},
\end{aligned}$$

where $\Sigma^d_{B_0}$ and $\Sigma^{od}_{B_0}$ are subbundles of $\Sigma_{B_0} \oplus \widehat{\Sigma}_{B_0}$ given by

$$\Sigma^d_{B_0} := \left\{ \begin{pmatrix} \varphi \\ \eta(\varphi) \end{pmatrix} : \varphi \in \Sigma_{B_0} \right\}, \qquad \Sigma^{od}_{B_0} := \left\{ \begin{pmatrix} \varphi \\ -\eta(\varphi) \end{pmatrix} : \varphi \in \Sigma_{B_0} \right\}.$$

The reason for this definition will become clear in the next section. Note that

$$\Gamma_{M_0}(\Sigma_{M_0}) = \bigoplus_{\mu\in\mathrm{Spec}_p(\mathrm{D}_{S^k})} W_\mu \quad \text{if } m \text{ even, or } m, k \text{ odd}, \tag{2.2}$$

$$\Gamma_{M_0}(\Sigma_{M_0}) = \bigoplus_{\mu\in\mathrm{Spec}_p(\mathrm{D}_{S^k})\cap\mathbb{R}^+} W_\mu \quad \text{if } m \text{ odd}, k \text{ even}. \tag{2.3}$$

3. The Dirac equation on the warped product

In this section we calculate the Dirac operator on M_0 in terms of the Dirac operator on B_0 and the fiber Dirac operator. For details concerning Dirac operators on Riemannian manifolds see [12] or [21].

First we recall some facts about the Levi-Civita connection ∇ on M_0. We denote the Levi-Civita connection on B_0 by ∇^{B_0} and by $\nabla^H : \Gamma T'M_0 \to \Gamma(T^*B \otimes T'M_0)$ the partial covariant derivative in the bundle of vertical vectors given by $\nabla^H_X Y := \nabla_X Y$ for $X \in TB_0$, $Y \in \Gamma T'M_0$. Furthermore, we denote by ∇' the partial covariant derivative in $T'M_0$ which is given by the Levi-Civita connection on the fibers, namely $\nabla'_{Y_1} Y_2 = \nabla^{f(b)S^k}_{Y_1} Y_2$ with $Y_1, Y_2 \in \Gamma T'_b M_0$. Therefore ∇^H and ∇' together define an isometric linear connection on $T'M$.

The second fundamental form II of the fibers is given by $\mathrm{II}(Y_i, Y_j) = -\delta_{ij} \frac{\operatorname{grad} f}{f}$ for orthonormal vertical vector fields Y_i, Y_j. Therefore the classical Gauss formula is

$$\nabla_{Y_i} Y_j = -\delta_{ij} \frac{\operatorname{grad} f}{f} + \nabla'_{Y_i} Y_j$$

for orthonormal $Y_i, Y_j \in \Gamma T'M_0$. From this we obtain, in the case m or k even, for the covariant derivative of a spinor $\psi = \pi_1^* \psi_{B_0} \otimes \psi'$ with $\psi_{B_0} \in \Gamma \Sigma_{B_0}$ and $\psi' \in \Gamma \Sigma'$,

$$\begin{aligned}
\nabla_X(\pi_1^*\psi_{B_0} \otimes \psi') &= \pi_1^* \nabla_X^{B_0} \psi_{B_0} \otimes \psi' + \pi_1^* \psi_{B_0} \otimes \nabla_X^H \psi' \quad \text{for } X \in \pi_1^* TB, \\
\nabla_Y(\pi_1^*\psi_{B_0} \otimes \psi') &= \pi_1^* \psi_{B_0} \otimes \nabla'_Y \psi' + \frac{1}{2} \frac{\operatorname{grad} f}{f} \cdot Y \cdot (\pi_1^* \psi_{B_0} \otimes \psi') \quad \text{for } Y \in \pi_2^* T\Sigma_{S^k}.
\end{aligned}$$

Here the induced covariant derivatives in the spinor bundle are denoted with the same symbol as the corresponding operators in the tangential bundle, i.e., $\nabla^{B_0} : \Gamma \Sigma_{B_0} \to \Gamma(T^*B_0 \otimes \Sigma_{B_0})$ is induced by the Levi-Civita connection on B_0 and $\nabla^H : \Gamma \Sigma' \to \Gamma(\pi_1^*(T^*B_0) \otimes \Sigma')$ is induced by the covariant derivative of the Levi-Civita connection on M_0 of vertical vectors with respect to horizontal vectors. The formulas for m and k odd are completely analogous. Therefore the Dirac operator D_{M_0} is given by

$$\begin{aligned}
\mathrm{D}_{M_0}(\pi_1^*\psi_B \otimes \psi') &= \pi_1^* \mathrm{D}_{B_0} \psi_B \otimes \psi' + \pi_1^*(\omega_{B_0} \cdot \psi_B) \otimes \mathrm{D}'\psi' + \frac{k}{2} \pi_1^* \left(\frac{\operatorname{grad} f}{f} \cdot \psi_B \right) \otimes \psi' \\
&\quad + \sum_{i=1}^m \pi_1^*(e_i \cdot \psi_B) \otimes \nabla^H_{e_i} \psi' \quad \text{for } m \text{ even}, \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}_{M_0}(\pi_1^*\psi_B \otimes \psi') &= \pi_1^* \mathrm{D}_{B_0} \psi_B \otimes \omega_{fS^k} \cdot \psi' + \pi_1^* \psi_B \otimes \mathrm{D}'\psi' + \frac{k}{2} \pi_1^* \left(\frac{\operatorname{grad} f}{f} \cdot \psi_B \right) \otimes \omega_{fS^k} \cdot \psi' \\
&\quad + \sum_{i=1}^m \pi_1^*(e_i \cdot \psi_B) \otimes \omega_{fS^k} \cdot \nabla^H_{e_i} \psi' \quad \text{for } m \text{ odd, } k \text{ even}, \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}_{M_0} \begin{pmatrix} \pi_1^*\psi_B \otimes \psi' \\ \pi_1^*\varphi_B \otimes \varphi' \end{pmatrix} &= \mathrm{i} \begin{pmatrix} \eta^{-1}(\mathrm{D}_{B_0}\varphi_B) \otimes \varphi' \\ \eta(\mathrm{D}_{B_0}\psi_B) \otimes \psi' \end{pmatrix} + \begin{pmatrix} \eta^{-1}(\varphi_B) \otimes \mathrm{D}'\varphi' \\ \eta(\psi_B) \otimes \mathrm{D}'\psi' \end{pmatrix} + \frac{\mathrm{i}k}{2} \begin{pmatrix} \eta^{-1}\left(\frac{\operatorname{grad} f}{f} \cdot \varphi_B\right) \otimes \varphi' \\ \eta\left(\frac{\operatorname{grad} f}{f} \cdot \psi_B\right) \otimes \psi' \end{pmatrix} \\
&\quad + \mathrm{i} \sum_{i=1}^m \begin{pmatrix} \eta^{-1}(e_i \cdot \varphi_B) \otimes \nabla^H_{e_i} \varphi' \\ \eta(e_i \cdot \psi_B) \otimes \nabla^H_{e_i} \psi' \end{pmatrix} \quad \text{for } m \text{ and } k \text{ odd}, \qquad (3.3)
\end{aligned}$$

where D_{B_0} is the Dirac operator on B_0 and $(e_1, \dots, e_m)$ is a local orthonormal basis of TB.

Theorem 3.1. *The Dirac operator on M_0 respects the decomposition* (2.2), (2.3) *of the space of spinors in spinors of weight μ,*

$$\mathrm{D}_{M_0} = \bigoplus_{\mu} \mathrm{D}_{\mu}$$

with $\mathrm{D}_\mu := \mathrm{D}_{M_0}|_{W_\mu}$ given as follows:

(1) *For m even*

$$\mathrm{D}_\mu = \begin{pmatrix} \mu/f & \mathrm{D}_B^+ \\ \mathrm{D}_B^- & -\mu/f \end{pmatrix}$$

with respect to the decomposition $W_\mu = \Gamma_{B_0}(\Sigma_{B_0}^+ \otimes \mathcal{E}_\mu) \oplus \Gamma_{B_0}(\Sigma_{B_0}^- \otimes \mathcal{E}_\mu)$.

(2) *For m odd and k even*

$$\mathrm{D}_\mu = \begin{pmatrix} \mu/f & \mathrm{D}_B \\ \mathrm{D}_B & -\mu/f \end{pmatrix}$$

on $W_\mu = \Gamma_{B_0}(\Sigma_{B_0} \otimes \mathcal{E}_\mu) \oplus \Gamma_{B_0}(\Sigma_{B_0} \otimes \mathcal{E}_{-\mu})$.

(3) *For m odd and k odd*

$$\mathrm{D}_\mu = \begin{pmatrix} \mu/f & \mathrm{i}\mathrm{D}_B \\ -\mathrm{i}\mathrm{D}_B & -\mu/f \end{pmatrix}$$

on $W_\mu = \Gamma_{B_0}(\Sigma_{B_0}^{d} \otimes \mathcal{E}_\mu) \oplus \Gamma_{B_0}(\Sigma_{B_0}^{od} \otimes \mathcal{E}_\mu)$, where here D_B denotes the sum of the Dirac operators on Σ_{B_0} and $\widehat{\Sigma}_{B_0}$.

Proof. We use the relation

$$\nabla^H_{e_i} \frac{\pi_1^* \psi_{S^k}}{f^{k/2}} = -\frac{k}{2} \frac{\mathrm{d}f(e_i)}{f} \frac{\pi_1^* \psi_{S^k}}{f^{k/2}}$$

for $\psi_{S^k} \in \Gamma \Sigma_{S^k}$. Then the representation in (1) follows from (3.1) and from $\omega_B|_{\Sigma_B^\pm} = \pm \mathrm{id}|_{\Sigma_B^\pm}$. The formula in (2) is a consequence of (3.2) and the fact that ω_{S^k} is an isomorphism from $E(\mu, \mathrm{D}_{S^k})$ to $E(-\mu, \mathrm{D}_{S^k})$. In the case (3) when m and k are odd we use the fact that the isomorphism η anti-commutes with the Dirac operators on B_0. Thus (3.3) gives

$$\mathrm{D}_{M_0}\left(\begin{pmatrix} \psi_B \\ \pm\eta(\psi_B) \end{pmatrix} \otimes \sigma'_{\mu,i} \right) = \mathrm{i} \begin{pmatrix} \mp \mathrm{D}_{B_0}\psi_B \\ \eta(\mathrm{D}_{B_0}\psi_B) \end{pmatrix} \otimes \sigma'_{\mu,i} + \begin{pmatrix} \pm\psi_B \\ \eta(\psi_B) \end{pmatrix} \otimes \frac{\mu}{f}\sigma'_{\mu,i}$$

where $\sigma'_{\mu,i}$ is defined as in (2.1). □

4. The boundary conditions

In the last two sections we have considered the space of spinors and the Dirac operator on $M_0 = M \setminus \partial B = B_0 \times_f S^k$. Obviously, not every spinor on M_0 defines a continuous spinor on M. In the following theorem we establish a necessary condition for a spinor to have a continuous extension to M.

Theorem 4.1. *Let ψ be a spinor of weight μ on M_0,*

$$\begin{aligned} &\psi \in \Gamma_{B_0}\big(\Sigma_{B^m} \otimes \mathbb{C}^{r(\mu)}\big) && \text{for } m \text{ even},\\ &\psi \in \Gamma_{B_0}\big(\Sigma_{B^m} \otimes \mathbb{C}^{2r(\mu)}\big) && \text{for } m \text{ odd}, k \text{ even},\\ &\psi \in \Gamma_{B_0}\big(\big(\Sigma^{d}_{B^m} \otimes \mathbb{C}^{r(\mu)}\big) \oplus \big(\Sigma^{od}_{B^m} \otimes \mathbb{C}^{r(\mu)}\big)\big) && \text{for } m \text{ and } k \text{ odd}. \end{aligned}$$

If ψ has a continuous extension to a spinor on M, then for each of the components ψ_i of ψ

$$\lim_{t\to 0} \frac{\psi_i(x,t)}{f^{k/2}(t)} \quad \text{exists for } (x,t) \in \partial B_g \times (0,\varepsilon). \tag{4.1}$$

Proof. Let U be a metric collar of ∂B, namely $U = \partial B_g \times [0,\varepsilon)$. Then we have $U \times_f S^k \cong \partial B \times D^{k+1}$. The restriction of a spinor $\psi \in \Gamma_{M_0}\Sigma_{M_0}$ to $(U \times_f S^k) \setminus \partial B$ is therefore a spinor over $\partial B \times (D^{k+1} \setminus \{0\})$. More exactly, in the case m and k even the image of

$$\psi = \pi_1^*(\theta_1\psi_{\partial B} + \theta_2\hat{\psi}_{\partial B}) \otimes \frac{\pi_2^*\sigma_i}{f^{k/2}} \in \pi_1^*\Sigma_{B_0} \otimes \pi_2^*\Sigma_{S^k}$$

with $\theta_i \in C^0(0,\varepsilon)$, $\psi_{\partial B} \in \Gamma\Sigma_{\partial B}$ and $\hat{\psi}_{\partial B} \in \Gamma\widehat{\Sigma}_{\partial B}$ under the isomorphism

$$\Sigma_{(U\times_f S^k)\setminus\partial B} \cong \big(\pi_3^*\Sigma_{\partial B} \otimes \pi_4^*\Sigma_{D^{k+1}\setminus\{0\}}\big) \oplus \big(\pi_3^*\widehat{\Sigma}_{\partial B} \otimes \pi_4^*\Sigma_{D^{k+1}\setminus\{0\}}\big)$$

is given by

$$\left(\pi_3^*\psi_{\partial B} \otimes \pi_4^*\theta_1\frac{\sigma_i}{f^{k/2}}\right) \oplus \left(\pi_3^*\hat{\psi}_{\partial B} \otimes \pi_4^*\theta_2\frac{\sigma_i}{f^{k/2}}\right)$$

and can be extended continuously to $U \times_f S^k$ if and only if

$$\theta_j\frac{\sigma_i}{f^{k/2}} \in \Gamma\Sigma_{D^{k+1}\setminus\{0\}} = C^0\big(D^{k+1}\setminus\{0\}, \mathbb{C}^{2^{k/2}}\big), \quad j = 1,2,$$

can be extended continuously to $\Gamma\Sigma_{D^{k+1}}$. Here π_3 and π_4 denote the projections on ∂B and D^{k+1}, respectively. The inclusion $S^k \hookrightarrow \mathbb{R}^{k+1}$ induces an inclusion $\Sigma_{S^k} \hookrightarrow \Sigma_{\mathbb{R}^{k+1}}$, and the image of σ_i under this inclusion is a nonzero map on S^k. Therefore a necessary condition for $\theta_j\frac{\sigma_i}{f^{k/2}}$ to be continuously extendable to D^{k+1} is that $\lim_{t\to 0}\frac{\theta_j}{f^{k/2}}$ exists. In all other cases the assertion is proved analogously. □

5. An operator theoretic theorem

In this section we present the operator theoretic theorem established in [20] by means of which we are going to prove eigenvalue estimates for the Dirac operator D_M. For the basic operator theoretic notions used in the following we refer the reader, e.g., to [16].

We consider a Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \widehat{\mathcal{H}}$ which is the orthogonal sum of two Hilbert spaces $\mathcal{H}$ and $\widehat{\mathcal{H}}$. We assume that $\widetilde{A}_0$ is an (unbounded) operator in $\widetilde{\mathcal{H}}$ which admits a block operator representation

$$\widetilde{A}_0 = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix}$$

with densely defined closable operators A, C and D acting between the respective spaces. The operator $\widetilde{A}_0$ is defined on its natural domain $(\mathcal{D}(A)\cap\mathcal{D}(C^*))\oplus(\mathcal{D}(C)\cap\mathcal{D}(D))$, and we suppose that this domain is dense in $\widetilde{\mathcal{H}}$ and that $\widetilde{A}_0$ is also closable. The closure of $\widetilde{A}_0$ is denoted by $\widetilde{A}$.

We recall that a point $z\in\mathbb{C}$ is called a point of regular type of $\widetilde{A}$ if there exists a constant $c_z>0$ such that

$$\big\|(\widetilde{A}-z)\tilde{x}\big\| \geqslant c_z\|\tilde{x}\|, \quad \tilde{x}\in\mathcal{D}(\widetilde{A}).$$

The set of points of regular type of $\widetilde{A}$ is denoted by $r(\widetilde{A})$. Obviously, if z is a point of regular type of $\widetilde{A}$, then $\widetilde{A}-z$ is injective and hence $z\notin \operatorname{Spec}_p(\widetilde{A})$.

Theorem 5.1. *Suppose the block operator matrix $\widetilde{A}_0$ is densely defined and closable with closure $\widetilde{A}$ and that there exist positive numbers α, δ such that*

$$\begin{aligned}&\Re(Ax,x) \geqslant \alpha\|x\|^2, && x\in\mathcal{D}(A)\cap\mathcal{D}(C^*),\\ &\Re(D\hat{x},\hat{x}) \leqslant -\delta\|\hat{x}\|^2, && \hat{x}\in\mathcal{D}(C)\cap\mathcal{D}(D).\end{aligned} \tag{5.1}$$

Then the strip $\{z\in\mathbb{C}\colon -\delta<\Re(z)<\alpha\}$ belongs to the set $r(\widetilde{A})$ of points of regular type of $\widetilde{A}$. In particular,

$$\operatorname{Spec}_p(\widetilde{A})\cap\{z\in\mathbb{C}\colon -\delta<\Re(z)<\alpha\}=\emptyset.$$

Proof. The theorem has been proved in [20] under the assumption that the operators A, C and D are closed (see Theorem 2.1 therein). The proof carries over without changes to the case that A, C and D are only closable, but we repeat it for the convenience of the reader.

Let $z\in\mathbb{C}$ be such that $-\delta<\Re(z)<\alpha$. In order to show that $z\in r(\widetilde{A})$, it is sufficient to prove that there exists a constant $c_z>0$ such that

$$\big\|(\widetilde{A}_0-z)\tilde{x}\big\| \geqslant c_z\|\tilde{x}\|, \quad \tilde{x}\in\mathcal{D}(\widetilde{A}_0). \tag{5.2}$$

For if $\tilde{y}\in\mathcal{D}(\widetilde{A})$ is arbitrary, then, since $\widetilde{A}$ is the closure of $\widetilde{A}_0$, there exists a sequence $(\tilde{x}_n)_1^\infty\subset\mathcal{D}(\widetilde{A}_0)$ such that $\tilde{x}_n\to\tilde{y}$, $\widetilde{A}_0\tilde{x}_n\to\widetilde{A}\tilde{y}$, $n\to\infty$ (see [16, Chapter III, § 5.3]). By (5.2), we know that $\|(\widetilde{A}_0-z)\tilde{x}_n\|\geqslant c_z\|\tilde{x}_n\|$, $n\in\mathbb{N}$. Taking the limit on both sides, we find that $\|(\widetilde{A}-z)\tilde{y}\|\geqslant c_z\|\tilde{y}\|$, that is, $z\in r(\widetilde{A})$.

Assume now that there exists no constant $c_z>0$ with (5.2). Then there exists a sequence $(\tilde{x}_n)_1^\infty\subset\mathcal{D}(\widetilde{A})$, $\|\tilde{x}_n\|=1$, such that $(\widetilde{A}_0-z)\tilde{x}_n\to 0$, $n\to\infty$. With $\tilde{x}_n=(x_n,\hat{x}_n)^t$ we obtain

$$(A-z)x_n+C\hat{x}_n\to 0, \quad C^*x_n+(D-z)\hat{x}_n\to 0, \quad n\to\infty,$$

which implies

$$\big((A-z)x_n,x_n\big)+(C\hat{x}_n,x_n)\to 0, \quad (C^*x_n,\hat{x}_n)+\big((D-z)\hat{x}_n,\hat{x}_n\big)\to 0, \quad n\to\infty.$$

If we form the difference of these relations it follows that

$$\big((A-z)x_n,x_n\big)-\big((D-z)\hat{x}_n,\hat{x}_n\big)+(C\hat{x}_n,x_n)-\overline{(C\hat{x}_n,x_n)}\to 0, \quad n\to\infty,$$

whence

$$\Re(Ax_n,x_n)-\Re(z)\|x_n\|^2-\Re(D\hat{x}_n,\hat{x}_n)+\Re(z)\|\hat{x}_n\|^2\to 0, \quad n\to\infty.$$

By the assumptions (5.1), the left-hand side is bounded from below by

$$\big(\alpha - \Re(z)\big)\|x_n\|^2 + \big(\delta + \Re(z)\big)\|\hat{x}_n\|^2,$$

which is strictly positive since $-\delta < \Re(z) < \alpha$ and $\|\tilde{x}_n\|^2 = \|x_n\|^2 + \|\hat{x}_n\|^2 = 1$, a contradiction. □

6. The Dirac operator on M

The Dirac operator D_M on a spin warped product M with boundary in the Hilbert space $L_2(M, \Sigma_M)$ with domain $\mathcal{D}(\mathrm{D}_M) = W_2^1(M, \Sigma_M)$ (the first order Sobolev space) is self-adjoint because M is a closed Riemannian manifold (see [25]).

Denote by $\Gamma_{\mathrm{ex}}^\infty(M_0, \Sigma_{M_0})$ the set of all C^∞-spinors on M_0 which possess a continuous extension to M. Then D_M is the closure of the restriction of the Dirac operator on M_0 to $\Gamma_{\mathrm{ex}}^\infty(M_0, \Sigma_{M_0})$:

$$\mathrm{D}_M = \overline{\mathrm{D}_{M_0}|_{\Gamma_{\mathrm{ex}}^\infty(M_0,\Sigma_{M_0})}}.$$

Hence $\mathrm{D}_{M_0}|_{\Gamma_{\mathrm{ex}}^\infty(M_0,\Sigma_{M_0})}$ is essentially self-adjoint.

Theorem 6.1. *For $\mu \in \mathrm{Spec}_p(\mathrm{D}_{S^k})$ we consider the Hilbert space $\widetilde{\mathcal{H}}_\mu = \mathcal{H}_\mu \oplus \widehat{\mathcal{H}}_\mu$ where*

$$\mathcal{H}_\mu \oplus \widehat{\mathcal{H}}_\mu = \begin{cases} L_2(\Sigma_B^+ \otimes \mathbb{C}^{r(\mu)}) \oplus L_2(\Sigma_B^- \otimes \mathbb{C}^{r(\mu)}) & \textit{for } m \textit{ even},\\ L_2(\Sigma_B \otimes \mathbb{C}^{r(\mu)}) \oplus L_2(\Sigma_B \otimes \mathbb{C}^{r(\mu)}) & \textit{for } m \textit{ odd}, k \textit{ even},\\ L_2(\Sigma_B^d \otimes \mathbb{C}^{r(\mu)}) \oplus L_2(\Sigma_B^{od} \otimes \mathbb{C}^{r(\mu)}) & \textit{for } m \textit{ and } k \textit{ odd}, \end{cases}$$

and for simplicity we write L_2 and W_2^1 for L_2-spaces and first order Sobolev spaces when the underlying space determined by $\mathcal{H}_\mu$ or $\widehat{\mathcal{H}}_\mu$ is clear. We introduce the operators

$$A_\mu : \mathcal{H}_\mu \to \mathcal{H}_\mu, \qquad \mathcal{D}(A_\mu) = \left\{\Psi_1 \in \mathcal{H}_\mu\colon \frac{1}{f}\Psi_1 \in L_2\right\}, \qquad A_\mu \Psi_1 = \frac{\mu}{f}\Psi_1,$$

$$\widehat{A}_\mu : \widehat{\mathcal{H}}_\mu \to \widehat{\mathcal{H}}_\mu, \qquad \mathcal{D}(\widehat{A}_\mu) = \left\{\Psi_2 \in \widehat{\mathcal{H}}_\mu\colon \frac{1}{f}\Psi_2 \in L_2\right\}, \qquad \widehat{A}_\mu \Psi_2 = \frac{\mu}{f}\Psi_2,$$

and

$$C : \widehat{\mathcal{H}}_\mu \to \mathcal{H}_\mu, \qquad \mathcal{D}(C) = \left\{\Psi_2 \in \widehat{\mathcal{H}}_\mu\colon \Psi_2 \in W_2^1, \Psi_2|_{\partial B} = 0\right\},$$

$$C\Psi_2 = \begin{cases} D_B^+ \Psi_2 & \textit{for } m \textit{ even},\\ D_B \Psi_2 & \textit{for } m \textit{ odd}, k \textit{ even},\\ \mathrm{i} D_B \Psi_2 & \textit{for } m \textit{ and } k \textit{ odd}. \end{cases}$$

Let $k > 1$. Then the operator $\mathrm{D}_{M_0}|_{\Gamma_{\mathrm{ex}}^\infty(M_0,\Sigma_{M_0})}$ has a representation

$$\mathrm{D}_{M_0}|_{\Gamma_{\mathrm{ex}}^\infty(M_0,\Sigma_{M_0})} = \left(\bigoplus_\mu \widetilde{A}_\mu\right)\Bigg|_{\Gamma_{\mathrm{ex}}^\infty(M_0,\Sigma_{M_0})} \tag{6.1}$$

with essentially self-adjoint operators $\widetilde{A}_\mu$ in $\widetilde{\mathcal{H}}_\mu$ which are given by a block operator representation

$$\widetilde{A}_\mu = \begin{pmatrix} A_\mu & C \\ C^* & -\widehat{A}_\mu \end{pmatrix}$$

with domain $\mathcal{D}(\widetilde{A}_\mu)$ *given by* $(\mathcal{D}(A_\mu)\cap\mathcal{D}(C^*))\oplus(\mathcal{D}(\widehat{A}_\mu)\cap\mathcal{D}(C))$, *i.e.*,

$$\left\{\Psi_1\in\mathcal{H}_\mu\colon\ \Psi_1\in W_2^1,\ \frac{1}{f}\Psi_1\in L_2\right\}\oplus\left\{\Psi_2\in\widehat{\mathcal{H}}_\mu\colon\ \Psi_2\in W_2^1,\ \Psi_2|_{\partial B}=0,\ \frac{1}{f}\Psi_2\in L_2\right\}.$$

Proof. By Theorem 4.1, the components of every spinor $\psi\in\Gamma^\infty_{\mathrm{ex}}(M_0,\Sigma_{M_0})$ satisfy the boundary condition (4.1). This boundary condition first implies that these components vanish at ∂B, i.e., the two components of $\psi=(\Psi_1,\Psi_2)^t$ fulfil Dirichlet boundary conditions on the basis manifold B. Secondly, it shows that, if $k>1$,

$$\int_0^\varepsilon\left|\frac{1}{f(t)}\Psi_i(x,t)\right|^2 g(t)\,\mathrm{d}t=\int_0^\varepsilon\left|\frac{\Psi_i(x,t)}{f^{k/2}(t)}\right|^2 f^{k-2}(t)g(t)\,\mathrm{d}t\leqslant c\int_0^\varepsilon f^{k-2}(t)\,\mathrm{d}t<\infty$$

with a constant $c>0$ uniformly in x, i.e., $\frac{1}{f}\Psi_i\in L_2$, $i=1,2$, due to the assumptions on the warpe functions g and f (see Definition 2.1 and Lemma 2.2). □

7. The eigenvalue estimate

Definition 7.1. An eigenvalue of the Dirac operator on a warped product with boundary is called *eigenvalue of weight* μ with $\mu\in\mathrm{Spec}_p(\mathrm{D}_{S^k})\cap\mathbb{R}^+$ if μ is the smallest positive number such that the corresponding eigenspace contains a spinor of weight μ or $-\mu$.

Theorem 7.2. *Let* $M=B\times_f S^k$ *be a warped product with boundary with* $k>1$. *Then the spectrum of the Dirac operator on* M *is symmetric with respect to* 0, *and if* λ *is an eigenvalue of weight* μ *we have the estimate*

$$|\lambda|\geqslant\frac{\mu}{f_{\max}},$$

where $f_{\max}:=\max\{f(x)\colon\ x\in B\}$. *In particular, the first positive eigenvalue* λ_0 *satisfies*

$$\lambda_0\geqslant\frac{k}{2f_{\max}}.$$

Proof. The first assertion is immediate if m is even and if m and k are odd. If m is odd and k is even, then if $(\Psi_{B,1},\Psi_{B,2})^t\in W_\mu=\Gamma((\Sigma_B\otimes\mathbb{C}^{r(\mu)})\oplus(\Sigma_B\otimes\mathbb{C}^{r(\mu)}))$ is an eigenspinor to the eigenvalue λ, then $(-\Psi_{B,2},\Psi_{B,1})^t\in W_\mu$ is an eigenspinor to the eigenvalue $-\lambda$.

For the second assertion we note that if λ is an eigenvalue of D_M, i.e., of the closure of the operator $\mathrm{D}_{M_0}|_{\Gamma^\infty_{\mathrm{ex}}(M_0,\Sigma_{M_0})}$, then it is also an eigenvalue of the operator

$$\mathcal{A}:=\bigoplus_\mu\widetilde{A}_\mu$$

in the representation (6.1). In particular, if λ is an eigenvalue of weight $\mu>0$ of D_M, then it is an eigenvalue of the operator $\widetilde{A}_\mu$. The operator $\widetilde{A}_\mu$ satisfies the assumptions of Theorem 5.1 with

$$\alpha=\delta=\frac{\mu}{f_{\max}}$$

since, for $\Psi^1 \in \mathcal{D}(A_\mu)$,

$$\big(A_\mu \Psi_1, \Psi_1\big)_{L_2} = \mu \int\limits_B \left|\frac{\Psi_1}{f}\right|^2 \mathrm{d}B \geqslant \frac{\mu}{f_{\max}} \|\Psi_1\|_{L_2}^2,$$

and analogously for $\widehat{A}_\mu$. Hence the second assertion ensues from Theorem 5.1. Finally, the last assertion follows from the fact that every eigenvalue of D_M is an eigenvalue of one of the operators $\widetilde{A}_\mu$ and

$$\min_\mu \big\{|\lambda|\colon\ \lambda \in \operatorname{Spec}_p(\widetilde{A}_\mu)\big\} \geqslant \min\left\{\frac{|\mu|}{f_{\max}}\colon\ \mu \in \operatorname{Spec}_p(\mathrm{D}_{S^k})\right\} \geqslant \frac{k}{2 f_{\max}},$$

where we have used that the smallest eigenvalue in modulus of the Dirac operator on the sphere S^k is $k/2$ (see [24]). □

In the following we show by means of an example that in general the lower bound for the first positive eigenvalue in the preceding theorem is sharp.

For this, let $L > 0$, and let D_L^m be the m-dimensional disc with radius L and $U_L^m := [L, L + \frac{\pi}{2}] \times_{\sin(t-L)+L} S^{m-1}$. Gluing these two manifolds, we set

$$B_L^m := D_L^m \amalg_{S_L^{m-1}} U_L^m.$$

Further, we define the warpe function

$$f : B_L^m \to \mathbb{R}$$

by

$$f|_{D_L^m} \equiv 1, \qquad f(t, x) := \cos(t - L), \quad (t, x) \in U_L^m.$$

On $M_L := B_L^m \times_f S^k$ we consider the isometrical immersion

$$\begin{aligned}
&M_L \to \mathbb{R}^{m+k+1},\\
&(b, y) \mapsto \big(b, \iota_k(y)\big), \quad b \in D_L^m,\ y \in S^k,\\
&(t, x, y) \mapsto \big(\big(\sin(t - L) + L\big)\iota_{m-1}(x), \cos(t - L)\iota_k(y)\big), \quad (t, x) \in U_L^m,\ y \in S^k,
\end{aligned}$$

where $\iota_r : S^r \to \mathbb{R}^{r+1}$ denotes the canonical immersion.

For an n-dimensional closed oriented hypersurface M isometrically immersed in $\mathbb{R}^{n+1}$ with mean curvature H in $\mathbb{R}^{n+1}$, the following estimate for the smallest eigenvalue in modulus of the Dirac operator on M holds (see [5,8]):

$$\lambda^2 \leqslant \frac{n^2 \int_M H^2 \,\mathrm{d}M}{4 \operatorname{vol} M}. \tag{7.1}$$

In the sequel, we calculate the upper bound (7.1) for the smallest positive eigenvalue of the Dirac operator D_{M_L} on M_L in the limit $L \to \infty$.

The mean curvature H on $U_L^m \times_f S^k$ depends only on $t \in [L, L + \pi/2]$, and for the function h given by $H(t, x) = h(t - L),\ t \in [L, L + \pi/2],\ x \in S^{m-1} \times S^k$, we have

$$h(t) = \frac{1}{m + k}\left(k + 1 + (m - 1)\frac{\sin t}{\sin t + L}\right).$$

Therefore the numerator of (7.1) is given by

$$(m+k)^2 \int_{M_L} H^2 \, \mathrm{d}M_L = (m+k)^2 \Biggl(\frac{k^2}{(m+k)^2} L^m \operatorname{vol} D^m \operatorname{vol} S^k + \int_0^{\pi/2} (\cos t)^k (\sin t + L)^{m-1} h^2(t) \, \mathrm{d}t \operatorname{vol} S^{m-1} \operatorname{vol} S^k \Biggr)$$
$$= k^2 L^m \operatorname{vol} D^m \operatorname{vol} S^k + \mathrm{O}\bigl(L^{m-1}\bigr),$$

and the denominator of (7.1) is given by

$$4 \operatorname{vol} M_L = 4L^m \operatorname{vol} D^m \operatorname{vol} S^k + \mathrm{O}\bigl(L^{m-1}\bigr).$$

The following remark, which we obtain from (7.1) together with the above calculations and from the fact that $f_{\max} = 1$ here, shows that the lower bounds of Theorem 7.2 are sharp in the limiting case $L \to \infty$.

Remark 7.3. The first positive eigenvalue $\lambda_0(L)$ of the Dirac operator on M_L satisfies the estimate

$$\lambda_0(L)^2 \leqslant \frac{k^2}{4} + \mathrm{O}\bigl(L^{-1}\bigr).$$

References

[1] B. Ammann, The Dirac operator on collapsing S^1-bundles, Preprint 39/1998, Math. Fak. Universität Freiburg.
[2] B. Ammann, Spin-Strukturen und das Spektrum des Dirac-Operators, Thesis, Shaker Verlag, Aachen, 1998.
[3] B. Ammann, C. Bär, The Dirac operator on nilmanifolds and collapsing circle bundles, Ann. Global Anal. Geom. 16 (1998) 221–253.
[4] C. Bär, The Dirac operator on space forms of positive curvature, J. Math. Soc. Japan 48 (1994) 69–83.
[5] C. Bär, Extrinsic bounds for eigenvalues of the Dirac operator, Ann. Glob. Anal. Geom. 16 (6) (1998) 573–596.
[6] C. Bär, Metrics with harmonic spinors, Geom. Funct. Anal. 6 (6) (1996) 899–942.
[7] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, Twistors and Killing Spinors on Riemannian Manifolds, Teubner Verlag, Stuttgart, 1991.
[8] M. Bordoni, Spectral estimates for the Schrödinger and Dirac type operators on Riemannian manifolds, Math. Ann. 298 (1994) 693–718.
[9] M. Cahen, A. Franc, S. Gutt, The spectrum of the Dirac operator on complex projective space $P_{2q-1}(\mathbb{C})$, Lett. Math. Phys. 18 (1989) 165–176.
[10] M. Cahen, A. Franc, S. Gutt, Erratum to: "The spectrum of the Dirac operator on complex projective space $P_{2q-1}(\mathbb{C})$", Lett. Math. Phys. 32 (1994) 365–368.
[11] M. Cahen, S. Gutt, A. Trautmann, Pin structures and the modified Dirac operator, J. Geom. Phys. 17 (1995) 283–297.
[12] Th. Friedrich, Dirac Operatoren in der Riemannschen Geometrie, Vieweg Verlag, Braunschweig, 1997.
[13] Th. Friedrich, Die Abhängigkeit des Dirac-Operators von der Spin-Struktur, Colloquium Mathematicum 48 (1) (1984) 57–62.
[14] Th. Friedrich, Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung, Math. Nachr. 97 (1980) 117–146.
[15] Th. Friedrich, E.C. Kim, The Einstein Dirac equation on Riemannian spin manifolds, J. Geom. Phys. 33 (2000) 128–172.
[16] T. Kato, Perturbation Theory for Linear Operators, Second Edition, Springer Verlag, Berlin, 1995.
[17] W. Kramer, Der Dirac Operator auf Faserungen, in: Bonner Mathematische Schriften 317, Bonn, 1999 (in German).

[18] M. Kraus, Lower bounds for eigenvalues of the Dirac operator on n-spheres with $SO(n)$-symmetry, J. Geom. Phys. 32 (2000) 341–348.
[19] M. Kraus, Eigenvalues of the Dirac operator on fibrations over S^1, Ann. Global Anal. Geom. 19 (3) (2001) 235–257.
[20] H. Langer, C. Tretter, Diagonalization of certain block operator matrices and applications to Dirac operators, Operator Theory: Adv. Appl. 122 (2001) 331–358.
[21] H.-B. Lawson, M.-L. Michelsohn, Spin Geometry, Princeton University Press, Princeton, 1989.
[22] L. Seeger, Metriken mit harmonischen Spinoren auf geradedimensionalen Sphären, Thesis, Shaker Verlag, Aachen, 2001.
[23] S. Seifarth, U. Semmelmann, The spectrum of the Dirac operator on the odd dimensional complex projective space $P^{2m-1}(\mathbb{C})$, SFB 288, Preprint 95, 1993.
[24] S. Sulanke, Berechnung des Spektrums des Quadrates des Dirac-Operators auf der Sphäre und Untersuchung zum ersten Eigenwert von D auf 5-dimensionalen Räumen konstanter positiver Schnittkrümmung, Thesis, Humboldt-Universität zu Berlin, 1981.
[25] J.A. Wolf, Essential self-adjointness for the Dirac operator and its square, Indiana Univ. Math. J. 22 (7) (1973) 611–640.